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Comments on the four-dimensional effective theory for warped compactification

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Abstract: We derive four-dimensional effective theories for warped compactification of the ten-dimensional IIB supergravity and the eleven-dimensional Hořava-Witten model. We show that these effective theories allow a much wider class of solutions than the original higher-dimensional theories. In particular, the effective theories have cosmological solutions in which the size of the internal space decreases with the cosmic expansion in the Einstein frame. This type of compactifying solutions are not allowed in the original higher-dimensional theories. This result indicates that the effective four-dimensional theories should be used with caution, if one regards the higher-dimensional theories more fundamental.

Keywords: Supergravity Models, Four-dimensional Effective Theory, Moduli Instability.

Contents

1. Introduction

Recently, a new class of dynamical solutions describing a size-modulus instability in the ten-dimensional type IIB supergravity model have been discovered by Gibbons et al. [1] and the authors [2]. These solutions can be always obtained by replacing the constant modulus h_0 in the warp factor $h = h_0 + h_1(y)$ for supersymmetric solutions by a linear function $h_0(x)$ of the four-dimensional coordinates x^μ. Such extensions exist for many of the well-known solutions compactified with flux on a conifold, resolved conifold, deformed conifold and compact Calabi-Yau manifold [2].

In most of the literature, the dynamics of the internal space, namely the moduli, in a higher-dimensional theory is investigated by utilising a four-dimensional effective theory. In particular, effective four-dimensional theories are used in essential ways in recent important work on the moduli stabilisation problem and the cosmological constant/inflation problem in the IIB sugra framework [3, 4, 5, 6, 7, 8]. Hence, it is

desirable to find the relation between the above dynamical solutions in the higher-dimensional theories and solutions in the effective four-dimensional theory.

In the conventional approach where the non-trivial warp factor does not exist or is neglected, an effective four-dimensional theory is derived from the original theory assuming the "product-type" ansatz for field variables [9, 10]. This ansatz requires that each basic field of the theory is expressed as the sum of terms of the form $f(x)\omega(y)$, where $f(x)$ is an unknown function of the four-dimensional coordinates x^μ, and $\omega(y)$ is a known harmonic tensor on the internal space. Further, it is assumed that the higher-dimensional metric takes the form $ds^2 = ds^2(\mathrm{X}_4) + h_0^\beta(x)ds^2(\mathrm{Y})$, where $ds^2(\mathrm{X}_4) = g_{\mu\nu}(x)dx^\mu dx^\nu$ is an unknown four-dimensional metric, $h_0(x)$ is the size modulus for the internal space depending only on the x-coordinates, and $ds^2(\mathrm{Y}) = \gamma_{pq}dy^p dy^q$ is a (Calabi-Yau) metric of the internal space that depends on the x-coordinates only through moduli parameters. Under this ansatz, the four-dimensional effective action is obtained by integrating out the known dependence on y^p in the higher-dimensional action.

The dynamical solutions in the warped compactification mentioned at the beginning, however, do not satisfy this ansatz. Hence, in order to incorporate such solutions to the effective theory, we have to modify the ansatz. Taking account of the structure of the supersymmetric solution, the most natural modification of the ansatz is to introduce the non-trivial warp factor h into the metric as $ds^2 = h^\alpha ds^2(\mathrm{X}_4) + h^\beta ds^2(\mathrm{Y})$ and assume that h depends on the four-dimensional coordinates x^μ only through the modulus parameter of the supersymmetric solution as in the case of the internal moduli degrees of freedom. This leads to the form $h = h_0(x) + h_1(y)$ for the IIB models, which is consistent with the structure of the dynamical solutions in the ten-dimensional theory.

In the present paper, starting from this modified ansatz, we study the dynamics of the four-dimensional effective theory and its relation to the original higher-dimensional theory for warped compactification of the ten-dimensional type IIB supergravity and the eleven-dimensional Hořava-Witten model. For simplicity, we assume that the moduli parameters other than the size parameter are frozen.

The paper is organised as follows. First, in the next section, we discuss the dynamics of the size modulus and the spacetime for compactification with vanishing flux in the ten-dimensional type IIB supergravity, starting from the standard "product-type" ansatz, for comparison. We show that the four-dimensional effective theory in this case is equivalent to the original ten-dimensional theory under the ansatz. Then, in the following two sections, we derive the four-dimensional effective theory for warped compactifications starting from the modified ansatz and compare it with the original higher-dimensional theory. The compactification on a compact Calabi-Yau manifold in

the ten-dimensional IIB supergravity is treated in §3, and the Hořava-Witten model of the eleven-dimensional heterotic M-theory is discussed in §4. In both of these models, it is shown that the four-dimensional effective theory contains spurious solutions that are not allowed in the original higher-dimensional theory. Finally, Section 5 is devoted to summary and discussion.

2. Compactification with vanishing flux in the 10D supergravity

When all form fluxes vanish and the dilaton is constant, the ten-dimensional supergravity reduces to the vacuum Einstein equations in ten dimensions, irrespective of the type of the theory. In this reduced theory, the direct product of the four-dimensional Minkowski spacetime and a six-dimensional Calabi-Yau space provides a supersymmetric solution. In this section, we briefly discuss the four-dimensional effective theory for this simple compactification, for comparison with the cases of flux compactification studied in the subsequent sections

2.1 Ansatz and a general solution

Let us consider the ten-dimensional spacetime with the metric

$$ds^2(\tilde{\mathrm{X}}_{10}) = h_0^{-1/2}(x)\, ds^2(\mathrm{X}_4) + h_0^{1/2}(x)\, ds^2(\mathrm{Y}_6), \tag{2.1}$$

where X_4 is the four-dimensional spacetime with coordinates x^μ, and Y_6 is the six-dimensional internal space. We assume that there exists no flux and the dilaton is constant. Then, if X_4 is flat, Y_6 is a Calabi-Yau manifold, and h_0 is a constant, this metric gives a supersymmetric solution to the ten-dimensional supergravity, and h_0 can be regarded as the parameter representing the size modulus of the internal space. Hence, when we discuss the four-dimensional dynamics of this size modulus, the metric (2.1) provides the most natural class, for which h_0 depends only on the coordinates x^μ of the four-dimensional spacetime and $ds^2(\mathrm{Y}_6)$ is some fixed metric on Y_6 that does not depend on x^μ .

Since we are assuming that all gauge fields vanish and the dilaton is constant, the dynamics is completely determined by the ten-dimensional vacuum Einstein equations, which read in the present case as

$$R_{\mu\nu}(\mathrm{X}_4) - h_0^{-1}\, D_\mu D_\nu h_0 + \frac{1}{4} g_{\mu\nu}(\mathrm{X}_4) h_0^{-1} \triangle_{\mathrm{X}} h_0 = 0, \tag{2.2a}$$

$$R_{pq}(\mathrm{Y}_6) - \frac{1}{4} g_{pq}(\mathrm{Y}_6) \triangle_{\mathrm{X}} h_0 = 0, \tag{2.2b}$$

where $g_{\mu\nu}(\mathrm{X}_4)$, $R_{\mu\nu}(\mathrm{X}_4)$, $\triangle_{\mathrm{X}}$ and D_μ denote the metric tensor, the Ricci tensor, the Laplacian, and the covariant derivative with respect to the metric $ds^2(\mathrm{X}_4)$, respectively,

and $g_{pq}(\mathrm{Y}_6)$ and $R_{pq}(\mathrm{Y}_6)$ denote the metric tensor and the Ricci tensor with respect to the metric $ds^2(\mathrm{Y}_6)$, respectively. Because $\triangle_{\mathrm{X}} h_0$ depends only on x^μ, and $R_{pq}(\mathrm{Y}_6)$ and $g_{pq}(\mathrm{Y}_6)$ depend only on the coordinates y^p of Y_6, (2.2b) requires that $\triangle_{\mathrm{X}} h_0$ is a constant. Hence, the equations (2.2) can be reduced to

$$R_{\mu\nu}(\mathrm{X}_4) = h_0^{-1}[D_\mu D_\nu h_0 - \lambda g_{\mu\nu}(\mathrm{X}_4)], \tag{2.3a}$$

$$R_{pq}(\mathrm{Y}_6) = \lambda g_{pq}(\mathrm{Y}_6), \tag{2.3b}$$

$$\triangle_{\mathrm{X}} h_0 = 4\lambda, \tag{2.3c}$$

where λ is a constant.

If we further assume that X_4 is Ricci flat, from (2.3a), the modulus h_0 is required to obey the equation

$$D_\mu D_\nu h_0 = \lambda g_{\mu\nu}(\mathrm{X}_4). \tag{2.4}$$

In the case of $(Dh_0)^2 \not\equiv 0$, this equation has a solution only when X_4 is locally flat, and its general solution for h_0 is given by

$$h_0(x) = \frac{\lambda}{2} x^\mu x_\mu + a_\mu x^\mu + b, \tag{2.5}$$

in terms of the four-dimensional Minkowski coordinates x^μ, where a_μ and b are constants satisfying the condition $a \cdot a \neq 0$. On the other hand, if $D_\mu h_0 \neq 0$ and $(D_\mu h_0)^2 = 0$, there exists a solution only when $\lambda = 0$, and a plane-wave-type geometry is also allowed for X_4 [2].

2.2 Four-dimensional effective theory

Next, we derive the four-dimensional effective theory for the four-dimensional spacetime and the size modulus in the setup of the previous subsection 2.1, i.e., under the assumptions that the ten-dimensional metric is given by (2.1), the dilaton is constant, and all form fluxes vanish. We also require that the internal space Y_6 has a fixed geometry satisfying (2.3b).

In this setup, the bosonic low-energy action for the ten-dimensional supergravity in the Einstein frame is simply given by the ten-dimensional Einstein-Hilbert action

$$S_{\mathrm{IIB}} = \frac{1}{2\tilde{\kappa}^2} \int_{\tilde{\mathrm{X}}_{10}} d\Omega(\tilde{\mathrm{X}}_{10}) R(\tilde{\mathrm{X}}_{10}), \tag{2.6}$$

where $\tilde{\kappa}$ is a positive constant. Here, under the assumption (2.1), the ten-dimensional scalar curvature $R(\tilde{\mathrm{X}}_{10})$ is expressed as

$$R(\tilde{\mathrm{X}}_{10}) = h_0^{1/2} R(\mathrm{X}_4) + h_0^{-1/2} R(\mathrm{Y}_6) - \frac{3}{2} h_0^{-1/2} \triangle_{\mathrm{X}} h_0\,, \tag{2.7}$$

where $R(\mathrm{X}_4)$ and $R(\mathrm{Y}_6)$ are the scalar curvatures of the metrics $ds^2(\mathrm{X}_4)$ and $ds^2(\mathrm{Y}_6)$, respectively. Inserting this expression into the action (2.6), we obtain the four-dimensional effective action

$$S_{\mathrm{IIB}} = \frac{1}{2\kappa^2}\int_{\mathrm{X}_4} d\Omega(\mathrm{X}_4)\left[h_0 R(\mathrm{X}_4) + 6\lambda\right], \tag{2.8}$$

where κ is given by $\kappa = (V_6)^{-1/2}\tilde{\kappa}$, V_6 is the volume of the internal space Y_6,

$$V_6 = \int_{\mathrm{Y}_6} d\Omega(\mathrm{Y}_6)\,, \tag{2.9}$$

and we have dropped the surface term coming from $\triangle_{\mathrm{X}} h_0$. It is easy to see that the four-dimensional Einstein equations and the field equation for h_0 obtained from this effective action are exactly identical to (2.3a) and (2.3c). Hence, the four-dimensional effective theory is equivalent to the original ten-dimensional theory under the ansatz adopted in this section.

Here, note that this effective theory has a kind of modular invariance when Y_6 is a flat torus or a Calabi-Yau space. To see this, by the conformal transformation $ds^2(\mathrm{X}_4) = h_0^{-1} ds^2(\bar{\mathrm{X}}_4)$, let us rewrite the four-dimensional effective action (2.8) in terms of the metric $\bar{g}_{\mu\nu}$ in the Einstein frame as

$$S_{\mathrm{IIB}} = \frac{1}{2\kappa^2}\int_{\bar{\mathrm{X}}_4} d\Omega(\bar{\mathrm{X}}_4)\left[R(\bar{\mathrm{X}}_4) - \frac{3}{2}(\bar{D}\ln h_0)^2 + 6\lambda h_0^{-2}\right], \tag{2.10}$$

where $R(\bar{\mathrm{X}}_4)$ and $\bar{D}_\mu$ are the Ricci scalar and the covariant derivative with respect to the metric $ds^2(\bar{\mathrm{X}}_4)$. It is easy to see that the action is invariant under the transformation $h_0 \to h_0^{-1}$, provided that $\lambda = 0$. Hence, if there is a solution for which h_0 increases in time, there is also a solution with the same four-dimensional metric in the Einstein frame such that h_0 decreases in time.

3. Flux compactification in the 10D IIB supergravity

In this section, we derive an effective theory describing the dynamics of the four-dimensional spacetime and the size modulus of the internal space for the flux compactification of the ten-dimensional type IIB supergravity. Then, we study the difference in the spacetime dynamics and the behavior of the size modulus for the four-dimensional effective theory and for the ten-dimensional theory.

3.1 Ten-dimensional solutions

In our previous work [2], we derived a general dynamical solution for warped compactification with fluxes in the ten-dimensional type IIB supergravity. In that work, we

imposed $d * (B_2 \wedge H_3) = 0$, which led to a slightly strong constraint on the free data for the solution, especially in the case of a compact internal space. Afterward, we have noticed that this condition is not necessary to solve the field equations, and without that condition, we can find a more general class of solutions. Because we take this class as the starting point of our argument, we first briefly explain how to get a general solution without that condition. We omit the details of calculations because they are essentially contained in our previous paper [2].

We assume that the ten-dimensional spacetime metric takes the form

$$ds^2(\tilde{\mathrm{X}}_{10}) = A(x,y)^2 ds^2(\mathrm{X}_4) + B(x,y)^2 ds^2(\mathrm{Y}_6), \tag{3.1}$$

where the meanings of $ds^2(\mathrm{X}_4)$ and $ds^2(\mathrm{Y}_6)$ and the other related notations are the same as in the previous section. $A(x,y)$ and $B(x,y)$ are arbitrary non-vanishing functions on $\tilde{\mathrm{X}}_{10}$ at the beginning. We further require that the dilaton and the form fields satisfy the following conditions:

$$\tau \equiv C_0 + i\, e^{-\Phi} = i g_s^{-1} (= \mathrm{const})\,, \tag{3.2a}$$

$$G_3 \equiv i g_s^{-1} H_3 - F_3 = \frac{1}{3!} G_{pqr}(y)\, dy^p \wedge dy^q \wedge dy^r\,, \tag{3.2b}$$

$$*_{\mathrm{Y}} G_3 = \epsilon i\, G_3 \quad (\epsilon = \pm 1)\,, \tag{3.2c}$$

$$\tilde{F}_5 = (1 \pm *) V_p dy^p \wedge \Omega(\mathrm{X}_4) = V \wedge \Omega(\mathrm{X}_4) \mp A^{-4} B^4 *_{\mathrm{Y}} V, \tag{3.2d}$$

where g_s is a constant representing the string coupling constant, and $*$ and $*_{\mathrm{Y}}$ are the Hodge duals with respect to the ten-dimensional metric $ds^2(\tilde{\mathrm{X}}_{10})$ and the six-dimensional metric $ds^2(\mathrm{Y}_6)$, respectively.

Under these assumptions, the two of the field equations,

$$\Box\tau + i\frac{(\nabla\tau)^2}{\tau_2} = -\frac{i}{2} G_3 \cdot G_3, \tag{3.3a}$$

$$*\tilde{F}_5 = \pm\tilde{F}_5, \tag{3.3b}$$

are automatically satisfied, and the rest are written

$$dG_3 = 0, \tag{3.4a}$$

$$\nabla \cdot G_3 = *d * G_3 = -i G_3 \cdot \tilde{F}_5, \tag{3.4b}$$

$$d\tilde{F}_5 = H_3 \wedge F_3, \tag{3.4c}$$

$$R_{MN} = \frac{g_s}{4}\left[\mathrm{Re}\,(G_{MPQ} G^{*\,PQ}_N) - \frac{1}{2} G_3 \cdot G_3^* g_{MN}\right] + \frac{1}{96} \tilde{F}_{MP_1\cdots P_4} \tilde{F}_N{}^{P_1\cdots P_4}. \tag{3.4d}$$

Among these equations, the first together with the assumptions (3.2b) and (3.2c) implies that G_3 is a closed imaginary-self-dual (ISD) 3-form on Y_6 that does not depend

on the coordinates x^μ. Then, (3.4b) can be rewritten as

$$\big(V \mp \epsilon d_y(A^4)\big) \cdot G_3 = 0, \tag{3.5}$$

where $d_y = dy^p \partial_p$. Since G_3 is an ISD form on Y_6, and V and $d_y(A^4)$ are 1-forms on Y_6, it follows from this equation that

$$V = \pm\epsilon d_y(A^4), \tag{3.6}$$

provided $G_3 \neq 0$.

Inserting this expression into (3.4c), we obtain the following two equations:

$$\partial_\mu(A^{-4}B^4\partial_p(A^4)) = 0, \tag{3.7a}$$

$$(\hat{D}\cdot(A^{-4}B^4\hat{D}(A^4))_{\mathrm{Y}} = \frac{g_s}{2}(G_3\cdot\bar{G}_3)_{\mathrm{Y}}, \tag{3.7b}$$

where $\hat{D}_p$ is the covariant derivative with respect to the metric $ds^2(\mathrm{Y}_6)$, and $(\alpha\cdot\beta)_{\mathrm{Y}}$ denotes the inner product of forms α and β on Y_6 with respect to the metric $ds^2(\mathrm{Y}_6)$.

Next, we consider the Einstein equations. First, from $R_{ap} = 0$ and (3.7a), we find that we can set

$$A = h(x,y)^{-1/4}, \quad B = h(x,y)^{1/4}, \tag{3.8}$$

by appropriately redefining $ds^2(\mathrm{X}_4)$ and $ds^2(\mathrm{Y}_6)$. Correspondingly, $\tilde{F}_5$ and (3.7b) can be written as

$$\tilde{F}_5 = \pm\epsilon(1\pm *)d(h^{-1})\wedge\Omega(\mathrm{X}_4), \tag{3.9}$$

$$\triangle_{\mathrm{Y}} h = -\frac{g_s}{2}(G_3\cdot\bar{G}_3)_{\mathrm{Y}}. \tag{3.10}$$

With these expressions, the ten-dimensional Einstein equations (3.4d) read

$$hR_{\mu\nu}(\mathrm{X}_4) - D_\mu D_\nu h + \frac{1}{4}g_{\mu\nu}(\mathrm{X}_4)\triangle_{\mathrm{X}} h = 0, \tag{3.11a}$$

$$\partial_\mu\partial_p h = 0, \tag{3.11b}$$

$$R_{pq}(\mathrm{Y}_6) - \frac{1}{4}g_{pq}(\mathrm{Y}_6)\triangle_{\mathrm{X}} h = 0. \tag{3.11c}$$

From the second of these equations, we immediately see that the warp factor h can be expressed as

$$h(x,y) = h_0(x) + h_1(y). \tag{3.12}$$

Further, if we require that $d_y h \neq 0$, the rest of the equations can be reduced to

$$R_{\mu\nu}(\mathrm{X}_4) = 0, \tag{3.13a}$$

$$D_\mu D_\nu h_0 = \lambda g_{\mu\nu}(\mathrm{X}_4), \tag{3.13b}$$

$$R_{pq}(\mathrm{Y}_6) = \lambda g_{pq}(\mathrm{Y}_6). \tag{3.13c}$$

Thus, we have found that the most general solutions satisfying the conditions (3.1) and (3.2) are specified by a Ricci flat spacetime X_4, an Einstein space Y_6, a closed ISD 3-form G_3 on Y_6, and the function $h(x,y)$ that is the sum of $h_0(x)$ satisfying (3.13b) and $h_1(x)$ satisfying (3.10). The additional constraint on G_3, $d_y[h^{-2}(B_2 \cdot dB_2)_{\mathrm{Y}}] = 0$, in Ref. [2] does not appear. Further, closed ISD 3-forms on Y_6 are in one-to-one correspondence with real harmonic 3-forms on Y_6. Hence, this class of dynamical solutions exist even for a generic compact Calabi-Yau internal space, if we allow $h_1(y)$ to be a singular function. This singular feature of h in the compact case with flux arises because h is a solution to the Poisson equation (3.10) and has nothing to do with the dynamical nature of the solution. It is shared by the other flux compactification models.

Here, note that the Ricci flatness of X_4 is required from the Einstein equations. This should be contrasted with the previous case with no warp. This point is quite important in the effective theory issue, as we see soon. Anyway, as explained in the previous section, the Ricci flatness of X_4 and (3.13b) are consistent only when X_4 is locally flat if $(Dh_0)^2 \neq 0$.

3.2 Four-dimensional effective theory

Now we study the four-dimensional effective theory that incorporates the dynamical solutions obtained in the previous subsection. For simplicity, we do not consider the internal moduli degrees of freedom of the metric of Y_6 or of the solution $h_1(y)$ in the present paper. Then, in its x-independent subclass with $\lambda = 0$, we have only one free parameter h_0. When we rescale $ds^2(\mathrm{Y}_6)$ by a constant ℓ as $\ell^2 ds^2(\mathrm{Y}_6) \to ds^2(\mathrm{Y}_6)$, we have to rescale h as $h/\ell^4 \to h$. We can easily see that the corresponding rescaled h_1 satisfies (3.10) again with the same G_3 as that before the rescaling. We can also confirm that the D3 brane charges associated with the 5-form flux do not change by this scaling. In contrast, h_0 changes its value by this rescaling. Therefore, h_0 represents the size modulus of the Calabi-Yau space Y_6.

From this observation, we construct the four-dimensional effective theory for the class of ten-dimensional configurations specified as follows. First, we assume that $\tilde{\mathrm{X}}_{10}$ has the metric

$$ds^2(\tilde{\mathrm{X}}_{10}) = h^{-1/2}(x,y)\, ds^2(\mathrm{X}_4) + h^{1/2}(x,y)\, ds^2(\mathrm{Y}_6), \tag{3.14}$$

where $h = h_0(x) + h_1(y)$ and $ds^2(\mathrm{Y}_6)$ is a fixed Einstein metric on Y_6 satisfying (3.13c), while $ds^2(\mathrm{X}_4)$ is an arbitrary metric on X_4. Further, we assume that the dilaton is frozen as in (3.2a), G_3 is given by a fixed closed ISD 3-form on Y_6, $h_1(y)$ is a fixed solution to (3.10), and $\tilde{F}_5$ is given by (3.9). Hence, the metric of X_4 and the function h_0 on it are the only dynamical variables in the effective theory.

The four-dimensional effective action for these variables can be obtained by evaluating the ten-dimensional action of the IIB theory

$$\begin{aligned} S_{\rm IIB} = & \frac{1}{2\tilde{\kappa}^2}\int_{\tilde{\rm X}_{10}} d\Omega(\tilde{\rm X}_{10})\left[R(\tilde{\rm X}_{10}) - \frac{\nabla_M\bar{\tau}\nabla^M\tau}{2({\rm Im}\,\tau)^2} - \frac{G_3\cdot\bar{G}_3}{2{\rm Im}\,\tau} - \frac{1}{4}\tilde{F}_5^2\right] \\ & \pm\frac{i}{8\tilde{\kappa}^2}\int_{\tilde{\rm X}_{10}}\frac{C_4\wedge G_3\wedge\bar{G}_3}{{\rm Im}\,\tau}, \end{aligned} \tag{3.15}$$

for the class of configurations specified above. In general, there is subtlety concerning the action of the type IIB supergravity, because the correct field equations can be obtained by imposing the self-duality condition (3.3b) after taking variation of the action in general. In the present case, however, since we are only considering configurations (3.9) satisfying the self-duality condition, this problem does not affect our argument. We can obtain the "correct" effective action by simply inserting (3.9) into the above ten-dimensional action.

First, for the metric (3.14) with $h = h_0(x) + h_1(y)$, the ten-dimensional scalar curvature $R(\tilde{\rm X}_{10})$ is expressed as

$$R(\tilde{\rm X}_{10}) = h^{1/2}R({\rm X}_4) + h^{-1/2}R({\rm Y}_6) - \frac{3}{2}h^{-1/2}\triangle_{\rm X}h_0 - \frac{1}{2}h^{-3/2}\hat{\triangle}_{\rm Y}h_1, \tag{3.16}$$

where $\triangle_{\rm X}$ and $\hat{\triangle}_{\rm Y}$ are the Laplacian with respect to the metrics $ds^2({\rm X}_4)$ and $ds^2({\rm Y}_6)$, respectively. Inserting this expression, (3.9), (3.10) and (3.13c) into (3.15), we get

$$\begin{aligned} S_{\rm IIB} = & \frac{1}{2\kappa^2}\int_{{\rm X}_4} d\Omega({\rm X}_4)\left[H(x)R({\rm X}_4) + 6\lambda + \frac{1}{2V_6}\int_{{\rm Y}_6} d\Omega({\rm Y}_6)\, h^{-1}\hat{\triangle}_{\rm Y}h_1\right] \\ & \pm\frac{i}{8\tilde{\kappa}^2}\int_{\tilde{\rm X}_{10}}\frac{C_{(4)}\wedge G_{(3)}\wedge\bar{G}_{(3)}}{{\rm Im}\tau}, \end{aligned} \tag{3.17}$$

where we have dropped the surface term coming from $\triangle_{\rm X}h_0$, $\kappa = (V_6)^{-1/2}\tilde{\kappa}$, and $H(x)$ is defined by

$$H(x) = h_0(x) + c; \quad c := V_6^{-1}\int_{{\rm Y}_6} d\Omega({\rm Y}_6)h_1. \tag{3.18}$$

The Chern-Simons term in this expression can be rewritten as follows. First, (3.10) can be written

$$i\epsilon g_s G_3\wedge\bar{G}_3 = 2d(\,*_{\rm Y}dh_1). \tag{3.19}$$

From this, it follows that

$$ig_sC_4\wedge G_3\wedge\bar{G}_3 = d(2\epsilon C_4\wedge\,*_{\rm Y}dh_1) \mp 2h^{-2}(dh_1\cdot dh_1)_{\rm Y}\,\Omega({\rm X}_4)\wedge\Omega({\rm Y}_6). \tag{3.20}$$

Hence, we have

$$\pm\frac{i}{8\tilde{\kappa}^2}\int_{\tilde{\mathrm{X}}_{10}}\frac{C_4\wedge G_3\wedge\bar{G}_3}{\mathrm{Im}\tau}=-\frac{1}{4V_6\kappa^2}\int_{\mathrm{X}_4}d\Omega(\mathrm{X}_4)\int_{\mathrm{Y}_6}d\Omega(\mathrm{Y}_6)\,\frac{\hat{\triangle}_{\mathrm{Y}}h_1}{h}$$
$$+\frac{1}{4\kappa^2V_6}\int_{\tilde{\mathrm{X}}_{10}}d\left[\left(\pm\epsilon C_4-h^{-1}\Omega(\mathrm{X}_4)\right)\wedge *_{\mathrm{Y}}dh_1\right]. \tag{3.21}$$

Note that apart from the boundary term, the contribution of Chern-Simons term is canceled by the term containing h_1 in (3.17), which came from the ten-dimensional scalar curvature and the 3-form flux. Consequently, neglecting the boundary term, we obtain the following four-dimensional effective action

$$S_{\mathrm{IIB}}=\frac{1}{2\kappa^2}\int_{\mathrm{X}_4}d\Omega(\mathrm{X}_4)\left[HR(\mathrm{X}_4)+6\lambda\right]. \tag{3.22}$$

This effective action has the same form as (2.8). Hence, it gives the four-dimensional field equations of the same form as in the no-flux case:

$$R_{\mu\nu}(\mathrm{X}_4)=H^{-1}\left[D_\mu D_\nu H-\lambda g_{\mu\nu}(\mathrm{X}_4)\right], \tag{3.23a}$$
$$\triangle_{\mathrm{X}}H=4\lambda. \tag{3.23b}$$

If the four-dimensional spacetime is Ricci flat, these equations reproduce the correct equation for $h_0(x)=H-c$ obtained from the ten-dimensional theory in the previous subsection. However, the Ricci flatness of X_4 is not required in the effective theory unlike in the ten-dimensional theory. Hence, the class of solutions allowed in the four-dimensional effective theory is much larger than the original ten-dimensional theory.

In particular, the effective theory has a modular invariance similar to that found in the no-flux Calabi-Yau case with $\lambda=0$. In fact, by the conformal transformation $ds^2(\mathrm{X}_4)=H^{-1}ds^2(\bar{\mathrm{X}}_4)$, (3.22) is expressed in terms of the variables in the Einstein frame as

$$S_{\mathrm{IIB}}=\frac{1}{2\kappa^2}\int_{\bar{\mathrm{X}}_4}d\Omega(\bar{\mathrm{X}}_4)\left[R(\bar{\mathrm{X}}_4)-\frac{3}{2}(\bar{D}\ln H)^2+6\lambda H^{-2}\right], \tag{3.24}$$

where $R(\bar{\mathrm{X}}_4)$ and $\bar{D}_\mu$ are the scalar curvature and the covariant derivative with respect to the metric $ds^2(\bar{\mathrm{X}}_4)$. The corresponding four-dimensional Einstein equations in the Einstein frame and the field equation for H are given by

$$R_{\mu\nu}(\bar{\mathrm{X}}_4)=\frac{3}{2}\bar{D}_\mu\ln H\,\bar{D}_\nu\ln H-3\lambda H^{-2}g_{\mu\nu}(\bar{\mathrm{X}}_4), \tag{3.25a}$$
$$\triangle_{\bar{\mathrm{X}}}\ln H=4\lambda H^{-2}, \tag{3.25b}$$

where $\triangle_{\bar{\mathrm{X}}}$ is the Laplacian with respect to the metric $ds^2(\bar{\mathrm{X}}_4)$. It is clear that for $\lambda = 0$, this action and the equations of motion are invariant under the transformation $H \to k/H$, where k is an arbitrary positive constant.

This transformation corresponds to the following transformation in the original ten-dimensional metric. Let us denote the new metric of X_4 and the function h obtained by this transformation by $ds'^2(\mathrm{X}_4)$ and h', respectively. Then, since the transformation preserves the four-dimensional metric in the Einstein frame, $ds'^2(\mathrm{X}_4)$ is related to $ds^2(\mathrm{X}_4)$ as $ds'^2(\mathrm{X}_4) = (H^2/k)ds^2(\mathrm{X}_4)$. In the meanwhile, from $H' = k/H = h'_0 + c$, h' is expressed in terms of the original h_0 as

$$h' = \frac{k}{h_0(x) + c} - c + h_1(y). \tag{3.26}$$

The corresponding ten-dimensional metric is written

$$ds^2 = k^{-1}H^2(h')^{-1/2}ds^2(\mathrm{X}_4) + (h')^{1/2}ds^2(\mathrm{Y}_6). \tag{3.27}$$

It is clear that this metric and h' do not satisfy the original ten-dimensional field equations. Hence, the modular-type invariance of the four-dimensional effective theory is not the invariance of the original ten-dimensional theory.

4. Hořava-Witten model in the 11D heterotic M-theory

A dynamical solution similar to that of the ten-dimensional IIB discussed in the previous section was found by Chen et al. [11] for the five-dimensional effective theory obtained from the Hořava-Witten model of the eleven-dimensional M-theory. In this section, we derive a four-dimensional effective theory for this five-dimensional theory.

4.1 Five-dimensional effective theory

We first briefly summarise the argument leading to the five-dimensional effective theory for the Hořava-Witten model of the eleven-dimensional M-theory. In this model, we first compactify the M-theory in eleven dimensions over $S^1/\mathbb{Z}_2$. Let the length of this compactifying circle S^1 be $2L$. Then, it is expected that E_8 gauge fields and their superpartners are created on the two orientifold planes to cancel the anomalies, leading to the $E_8 \times E_8$ heterotic theory in ten dimensions in the limit of small L. Hence, the action of the Hořava-Witten model is given by [12, 13]

$$\begin{aligned} S_{\mathrm{HW}} = {} & \frac{1}{2\hat{\kappa}^2}\int_{\hat{\mathrm{X}}_{11}} d\Omega(\hat{\mathrm{X}}_{11})\left[R(\hat{\mathrm{X}}_{11}) - \frac{1}{2}F_4^2\right] - \frac{1}{12\hat{\kappa}^2}A_3 \wedge F_4 \wedge F_4 \\ & - \frac{1}{8\pi\hat{\kappa}^2}\left(\frac{\hat{\kappa}}{4\pi}\right)^{2/3}\sum_{j=1,2}\int_{\mathrm{X}_{10}^{(j)}} d\Omega(\mathrm{X}_{10})\left[\mathrm{tr}(F^{(j)})^2 - \frac{1}{2}\mathrm{tr}R^2\right], \end{aligned} \tag{4.1}$$

where $\hat{\kappa}$ is the positive constant, $R(\hat{\mathrm{X}}_{11})$ is the scalar curvature with respect to the eleven-dimensional metric $ds^2(\hat{\mathrm{X}}_{11})$, A_3 is the 3-form gauge field with the field strength $F_4 = dA_3$, and $F^{(1)}$ and $F^{(2)}$ are the E_8 gauge field strengths. We choose the range $-L \le z \le L$ for the coordinate of S^1 with the end points being identified and impose the $\mathbb{Z}_2$ symmetry under the transformation $z \to -z$. The orientifold planes of this transformation, $\mathrm{X}_{10}^{(i)} (i = 1, 2)$, correspond to $z = 0$ and $z = L$. For simplicity, we will not consider the boundary gauge fields in the present paper.

If we further compactify this model over a six-dimensional internal space Y_6, then we obtain a four-dimensional model. In practice, the argument becomes simpler if we reverse the order of compactifications, i.e., if we compactify the M-theory first over Y_6 to $\hat{\mathrm{X}}_{11} = \tilde{\mathrm{X}}_5 \times \mathrm{Y}_6$ and then over $S^1/\mathbb{Z}_2$ to $\tilde{\mathrm{X}}_5 = \mathrm{X}_4 \times S^1/\mathbb{Z}_2$, as was done by Lukas et al [14]. In the first step, we obtain an effective five-dimensional theory. At this step, we assume that the eleven-dimensional metric takes the form

$$ds^2(\hat{\mathrm{X}}_{11}) = e^{2\phi(\tilde{x})/3} ds^2(\tilde{\mathrm{X}}_5) + e^{-\phi(\tilde{x})/3} ds^2(\mathrm{Y}_6), \tag{4.2}$$

where $\tilde{x}^a$ are the coordinates of the five-dimensional spacetime $\tilde{\mathrm{X}}_5$, and that the 4-form flux is expressed as

$$F_4 = (\omega \cdot \Omega(\mathrm{Y}_6))_{\mathrm{Y}} , \tag{4.3}$$

where ω is a 2-form on Y_6. We can show that even if we start from a more general warped metric of the form $ds^2(\hat{\mathrm{X}}_{11}) = e^{\alpha} ds^2(\tilde{\mathrm{X}}_5) + e^{\beta} ds^2(\mathrm{Y}_6)$, the field equations require both α and β to depend either only on $\tilde{x}^a$ or only on y^p, if F_4 takes the form (4.3). Hence, the above choice for the metric form is quite natural when we study dynamical instability of supersymmetric solutions in the Hořava-Witten model.

From the field equations

$$dF_4 = 0, \quad d*F_4 + \frac{1}{2} F_4 \wedge F_4 = 0, \tag{4.4}$$

we obtain

$$\partial_a \omega = 0, \quad d\omega = 0, \quad \hat{D} \cdot \hat{\omega} = 0, \tag{4.5}$$

where $\hat{\omega}$ is a 2-form on Y_6 such that $\hat{\omega}_{pq} = \omega_{pq}$ and their indices are raised and lowered by the metric $ds^2(\mathrm{Y}_6)$. Next, the traceless part of the Einstein equations for R_{pq} gives

$$R_{pq}(\mathrm{Y}_6) - \frac{1}{6} R(\mathrm{Y}_6) \hat{g}_{pq}(\mathrm{Y}_6) = -\frac{e^{\phi}}{2} \left(\hat{\omega}_{pr} \hat{\omega}_q{}^r - \frac{1}{3} \hat{\omega}^2 \hat{g}_{pq} \right). \tag{4.6}$$

From this, it follows that if $\partial_a \phi \neq 0$, both sides of this equation should vanish separately. Hence, taking account of the Bianchi identity, we obtain

$$R_{pq}(\mathrm{Y}_6) = \lambda g_{pq}(\mathrm{Y}_6), \tag{4.7}$$

$$\hat{\omega}_{pr} \hat{\omega}_q{}^r = \frac{1}{3} \hat{\omega}^2 g_{pq}(\mathrm{Y}_6). \tag{4.8}$$

Inserting these relations to the R_p^p equation, we have

$$6\lambda + e^{-\phi}\tilde{D}^2\phi = e^{\phi}\hat{\omega}^2, \tag{4.9}$$

from which and the rest of the Einstein equations, we obtain the constraint $\hat{\omega}^2 = 2m^2 =$ const and the field equations in the five-dimensional theory

$$R_{ab}(\tilde{\mathrm{X}}_5) = \frac{1}{2}\partial_a\phi\partial_b\phi + \left(\frac{m^2}{3}e^{2\phi} - 2\lambda e^{\phi}\right) g_{ab}(\tilde{\mathrm{X}}_5), \tag{4.10a}$$

$$\Box_{\tilde{\mathrm{X}}}\phi - 2m^2 e^{2\phi} = -6\lambda e^{\phi}, \tag{4.10b}$$

where $\Box_{\tilde{\mathrm{X}}}$ is the D'Alermbertian for the five-dimensional metric $ds^2(\tilde{\mathrm{X}}_5)$.

These field equations can be obtained from the five-dimensional effective action given by [11, 14]

$$S_{\mathrm{HW}} = \frac{1}{2\tilde{\kappa}^2}\int_{\tilde{\mathrm{X}}_5} d\Omega(\tilde{\mathrm{X}}_5)\left[R(\tilde{\mathrm{X}}_5) - \frac{1}{2}(\tilde{D}\phi)^2 - m^2 e^{2\phi} + 6\lambda e^{\phi}\right], \tag{4.11}$$

where $\tilde{\kappa} = (V_6)^{-1/2}\hat{\kappa}$, V_6 is the volume of Y_6.

4.2 Four-dimensional effective theory

In the Hořava-Witten model, a four-dimensional theory is obtained from the five-dimensional theory by compactification over $S^1/\mathbb{Z}_2$. Without loss of generality, the metric obtained by this compactification can be written $ds^2 = e^{\gamma}ds^2(\mathrm{X}_4) + e^{\delta}dz^2$. In general, the field equations do not lead to no relation between the warp factors e^{γ} and e^{δ} in this theory, and there exists no natural reduction to four dimensions. Hence, in order to obtain a four dimensional reduction, we have to impose some relation between e^{γ} and e^{δ}. In the present paper, to include the dynamical solution found by Chen et al. [11], we adopt the ansatz that $ds^2(\tilde{\mathrm{X}}_5)$ can be written

$$ds^2(\tilde{\mathrm{X}}_5) = h^{1/2}(x,z)\, ds^2(\mathrm{X}_4) + h(x,z)\, dz^2, \tag{4.12}$$

and the warp factor h has the structure

$$h(x,z) = h_0(x) + h_1(z). \tag{4.13}$$

We also assume that Y_6 is a Calabi-Yau space, i.e. $\lambda = 0$. As is shown in Appendix A, the most general solution to the field equations (4.10) satisfying this ansatz and the conditions $\partial_\mu h_0 \neq 0$ and $\partial_z h_1 \neq 0$ is given by

$$R_{\mu\nu}(\mathrm{X}_4) = 0, \qquad h(x,z) = h_0(x) + kz, \qquad e^{2\phi} = h^{-3}, \tag{4.14}$$

where $k^2 = 8m^2/3$, and h_0 is a solution to

$$D_\mu D_\nu h_0 = 0. \tag{4.15}$$

In the case $(Dh_0)^2 \neq 0$, which requires that X_4 is locally flat [2], this solution (4.14) is identical to the solution found by Chen et al. [11] (See Appendix A).

On the basis of this result, we construct a four-dimensional effective theory of the Hořava-Witten model for the class of five-dimensional configurations in which the metric is expressed as (4.12) with h of the form (4.13), and ϕ is related to h by

$$\phi = -\frac{3}{2}\ln h. \tag{4.16}$$

For this class of configurations, the five-dimensional action (4.11) can be written

$$S_{\mathrm{HW}} = \frac{1}{2\tilde{\kappa}^2}\int_{\mathrm{X}_4} d\Omega(\mathrm{X}_4)\int_0^L dz\left[h\,R(\mathrm{X}_4) - \frac{2\partial_z^2 h_1}{h^{1/2}} + \frac{5(\partial_z h_1)^2}{8h^{3/2}} - \frac{3k^2}{8h^{3/2}}\right]. \tag{4.17}$$

In order to perform the integration over z, we have to specify $h_1(z)$. In the present case, the only possible choice is

$$h_1(z) = kz; \quad k^2 = \frac{8m^2l^2}{3}. \tag{4.18}$$

However, the simple insertion of this expression into the above action does not give a correct result. This is because the variation of the action (4.17) with respect to h produces boundary terms at the orientifold planes $z = 0,\, L$, which do not vanish for the above choice of h_1. By inspecting the structure of these boundary terms, we find that if we add the additional term to the action given by

$$S_{\mathrm{boundary}} = \frac{1}{2\tilde{\kappa}^2}\int_{\mathrm{X}_4} d\Omega(\mathrm{X}_4)\left[\frac{1}{2}h^{-1/2}k\right]_{z=0}^{z=L}, \tag{4.19}$$

the correct field equations are obtained in five-dimension. Therefore, the four-dimensional effective action is given by

$$S \equiv S_{\mathrm{HW}} + S_{\mathrm{boundary}} = \frac{1}{2\kappa^2}\int_{\mathrm{X}_4} d\Omega(\mathrm{X}_4)H(x)R(\mathrm{X}_4),$$

where $\kappa = (L)^{-1/2}\tilde{\kappa}$, and $H(x)$ is defined by

$$H(x) = h_0(x) + \frac{kL}{2}. \tag{4.20}$$

Thus, we have obtained the same four-dimensional effective action as in the case of the type IIB supergravity in ten dimensions. In particular, the four-dimensional effective theory of the Hořava-Witten model allows solutions that cannot be uplifted to solutions in five dimensions or in eleven dimensions and has the same modular invariance as in the previous case, which is not respected in the original higher-dimensional theory, with respect to the size modulus in the Einstein frame.

5. Conclusion

In the present paper, we have derived four-dimensional effective theories for the spacetime metric and the size modulus of the internal space for warped compactification with flux in the ten-dimensional type IIB supergravity and in the Hořava-Witten model of the eleven-dimensional M-theory. The basic idea was to consider field configurations in higher dimensions that are obtained by replacing the constant size modulus in supersymmetric solutions for warped compactifications, by a field on the four-dimensional spacetime. The effective action for this moduli field and the four-dimensional metric has been determined by evaluating the higher-dimensional action for such configurations. In all cases, the dynamical solutions in the ten- and eleven-dimensional theories found by Gibbons et al. [1], Kodama and Uzawa [2] and Chen et al. [11] were reproduced in the four-dimensional effective theories.

In addition to this, we have found that these four-dimensional effective theories have some unexpected features. First, the effective actions of both theories are exactly identical to the four-dimensional effective action for direct-product type compactifications with no flux in ten-dimensional supergravities. In particular, the corresponding effective theory has a kind of modular invariance with respect to the size modulus field in the Einstein frame. This implies that if there is a solution in which the internal space expands with the cosmic expansion, there is always a conjugate solution in which the internal space shrinks with the cosmic expansion.

Second, the four-dimensional effective theory for warped compactification allows solutions that cannot be obtained from solutions in the original higher-dimensional theories. The modular invariance in the four-dimensional theory mentioned above is not respected in the original higher-dimensional theory either. This situation should be contrasted with the no-warp case in which the four-dimensional effective theory and the original higher-dimensional theory are equivalent under the product-type ansatz for the metric structure. This result implies that we have to be careful when we use a four-dimensional effective theory to analyse the moduli stabilisation problem and the cosmological problems in the framework of warped compactification of supergravity or M-theory.

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Appendix

A. Solutions of the 5D Hořava-Witten model

In this appendix, we prove that the solutions specified by (4.14) and (4.15) exhaust all solutions to the field equations (4.10) in the five-dimensional Hořava-Witten theory, if we assume that the five-dimensional metric takes the form

$$ds^2(\tilde{\mathrm{X}}_5) = h^{1/2}\, ds^2(\mathrm{X}_4) + h\, dz^2, \tag{A.1}$$

with

$$h = h_0(x) + h_1(z); \qquad \partial_\mu h_0 \neq 0, \quad \partial_z h_1 \neq 0. \tag{A.2}$$

For the metric (A.1), the field equations (4.10) can be written

$$D\cdot(hD\phi) + \partial_z(h^{1/2}\partial_z\phi) = 2m^2 h^{3/2} e^{2\phi}, \tag{A.3a}$$

$$R_{\mu\nu}(\mathrm{X}_4) - \frac{1}{4}g_{\mu\nu}(\mathrm{X}_4)R(\mathrm{X}_4) + \frac{9}{8h^2}\left[D_\mu h D_\nu h - \frac{1}{4}(Dh)^2 g_{\mu\nu}(\mathrm{X}_4)\right]$$
$$-\frac{1}{h}\left[D_\mu D_\nu h - \frac{1}{4}\triangle_{\mathrm{X}} h g_{\mu\nu}(\mathrm{X}_4)\right] = \frac{1}{2}\left[D_\mu\phi D_\nu\phi - \frac{1}{4}(D\phi)^2 g_{\mu\nu}(\mathrm{X}_4)\right], \tag{A.3b}$$

$$R(\mathrm{X}_4) - \frac{2\triangle_{\mathrm{X}} h}{h} + \frac{9(Dh)^2}{8h^2} - \frac{\partial_z^2 h}{h^{3/2}} + \frac{(\partial_z h)^2}{2h^{5/2}} = \frac{1}{2}(D\phi)^2 + \frac{4}{3}m^2 e^{2\phi} h^{1/2}, \tag{A.3c}$$

$$-\frac{3}{4}h^{1/2} D_\mu\left(\frac{\partial_z h}{h^{3/2}}\right) = \frac{1}{2}\partial_\mu\phi\partial_z\phi, \tag{A.3d}$$

$$-\frac{\triangle_{\mathrm{X}} h}{2h^{1/2}} - \frac{\partial_z^2 h}{h} + \frac{5(\partial_z h)^2}{4h^2} = \frac{1}{2}(\partial_z\phi)^2 + \frac{1}{3}m^2 e^{2\phi} h, \tag{A.3e}$$

where $R(\mathrm{X}_4)$, $R_{\mu\nu}(\mathrm{X}_4)$, $\triangle_{\mathrm{X}}$ and D_μ are the scalar curvature, the Ricci tensor, the Laplacian and the covariant derivative with respect to the metric $ds^2(\mathrm{X}_4)$.

First, from the assumption (A.2), (A.3d) reduces to

$$\partial_\mu\phi = \frac{9\partial_z h_1}{4h^2\partial_z\phi}\partial_\mu h_0. \tag{A.4}$$

Under the condition $\partial_z h_1 \neq 0$, this equation is equivalent to

$$\phi = \Phi(h_0, h_1), \qquad \Phi_0\Phi_1 = \frac{9}{4h^2}, \tag{A.5}$$

where $\Phi_0 \equiv \partial_{h_0}\Phi$ and $\Phi_1 \equiv \partial_{h_1}\Phi$.

With the help of these relations, (A.3b) can be written

$$R_{\mu\nu}(\mathrm{X}_4) - \frac{1}{4}g_{\mu\nu}(\mathrm{X}_4)R(\mathrm{X}_4) + \left(\frac{9}{8h^2} - \frac{1}{2}\Phi_0^2\right)\left[D_\mu h_0 D_\nu h_0 - \frac{1}{4}(Dh_0)^2 g_{\mu\nu}(\mathrm{X}_4)\right]$$
$$-\frac{1}{h}\left[D_\mu D_\nu h_0 - \frac{1}{4}\triangle_{\mathrm{X}} h_0 g_{\mu\nu}(\mathrm{X}_4)\right] = 0. \tag{A.6}$$

Differentiating this equation by y, we get

$$\left(\frac{9}{4h}+h^2\Phi_0\Phi_{01}\right)\left[D_\mu h_0 D_\nu h_0-\frac{1}{4}(Dh_0)^2 g_{\mu\nu}(\mathrm{X}_4)\right]=D_\mu D_\nu h_0-\frac{1}{4}\triangle_{\mathrm{X}} h_0 g_{\mu\nu}(\mathrm{X}_4), \quad (A.7)$$

where $\Phi_{01}\equiv\partial_{h_0}\partial_{h_1}\Phi$. The factor in the square bracket on the left-hand side of this equation does not vanish under the condition $\partial_\mu h_0\neq 0$ because of the regularity of $g_{\mu\nu}$ as a matrix, and the right-hand side does not depend on z. Hence, the first factor on the left-hand side should be independent of z:

$$0=\partial_{h_1}\left(\frac{9}{4h}+h^2\Phi_0\Phi_{01}\right)=\frac{9}{4}\partial_{h_0}\partial_{h_1}\ln(h\Phi_1). \quad (A.8)$$

Solving this with respect to Φ_1 and using (A.5), we obtain

$$\Phi_0=\frac{9}{4ha(h_0)b(h_1)}, \quad \Phi_1=\frac{a(h_0)b(h_1)}{h}. \quad (A.9)$$

The consistency of these equations, $\partial_{h_1}\Phi_0=\partial_{h_0}\Phi_1$, leads to

$$\frac{4}{9}\left[-a^2+ha\left(\partial_{h_0}a\right)\right]+\frac{\partial_{h_1}b}{b^3}h+\frac{1}{b^2}=0. \quad (A.10)$$

Differentiating this equation by h_0 yields

$$\frac{4}{9}a\left(\partial_{h_0}a\right)+h_0\partial_{h_1}\left(\frac{\partial_{h_1}b}{b^3}\right)-\frac{\partial_{h_1}b}{b^3}+h_1\partial_{h_1}\left(\frac{\partial_{h_1}b}{b^3}\right)=0. \quad (A.11)$$

This equation implies that $a(\partial_{h_0}a)$ is a linear function of h_0. Hence, we get

$$a^2=ph_0^2+2qh_0+s, \qquad \frac{1}{b^2}=\frac{4}{9}\left(ph_1^2-2qh_1+s\right), \quad (A.12)$$

where p,q and s are constant parameters. Inserting these expressions into (A.9), we obtain

$$\Phi_0=\pm\frac{3}{2h}\sqrt{\frac{ph_1^2-2qh_1+s}{ph_0^2+2qh_0+s}}, \quad \Phi_1=\pm\frac{3}{2h}\sqrt{\frac{ph_0^2+2qh_0+s}{ph_1^2-2qh_1+s}}. \quad (A.13)$$

This can be integrated to yield

$$e^{\mp 2\phi/3}=\frac{c}{h}\left[-ph_0h_1+q(h_0-h_1)+s+\sqrt{ph_0^2+2qh_0+s}\sqrt{ph_1^2-2qh_1+s}\right], \quad (A.14)$$

where c is a constant.

Using this expression for ϕ, (A.6) can be rewritten as

$$h\left[R_{\mu\nu}(\mathrm{X}_4)-\frac{1}{4}g_{\mu\nu}(\mathrm{X}_4)R(\mathrm{X}_4)\right]-D_\mu D_\nu h_0+\frac{1}{4}\triangle_{\mathrm{X}}h_0 g_{\mu\nu}(\mathrm{X}_4)$$
$$+\frac{9}{8}\frac{p(h_0-h_1)+2q}{ph_0^2+2qh_0+s}\left[D_\mu h_0 D_\nu h_0-\frac{1}{4}(Dh_0)^2 g_{\mu\nu}(\mathrm{X}_4)\right]=0. \tag{A.15}$$

Note that the left-hand side of this equation depends on h_1 linearly. Thus, this equation can be decomposed into two equations

$$R_{\mu\nu}(\mathrm{X}_4)-\frac{1}{4}g_{\mu\nu}(\mathrm{X}_4)R(\mathrm{X}_4)=\frac{9}{8}\frac{p}{ph_0^2+2qh_0+s}\left[D_\mu h_0 D_\nu h_0-\frac{1}{4}(Dh_0)^2 g_{\mu\nu}(\mathrm{X}_4)\right], \tag{A.16a}$$
$$D_\mu D_\nu h_0-\frac{1}{4}\triangle_{\mathrm{X}}h_0 g_{\mu\nu}(\mathrm{X}_4)=\frac{9}{4}\frac{ph_0+q}{ph_0^2+2qh_0+s}\left[D_\mu h_0 D_\nu h_0-\frac{1}{4}(Dh_0)^2 g_{\mu\nu}(\mathrm{X}_4)\right]. \tag{A.16b}$$

Multiplying the second of these by $D^\nu h_0$, we obtain

$$D_\mu(Dh_0)^2-\frac{1}{2}(\triangle_{\mathrm{X}}h_0)D_\mu h_0=\frac{27}{16}\frac{ph_0+q}{ph_0^2+2qh_0+s}(Dh_0)^2 D_\mu h_0. \tag{A.17}$$

From this, we find that if h_0 satisfies $(Dh_0)^2=0$, then $\triangle_{\mathrm{X}}h_0=0$ holds. On the other hand, if $(Dh_0)^2\neq 0$, this equation can be deformed as

$$D_\mu\left[\ln(Dh_0)^2-\frac{27}{32}\ln(ph_0^2+2qh_0+s)\right]=\frac{\triangle_{\mathrm{X}}h_0}{2(Dh_0)^2}D_\mu h_0. \tag{A.18}$$

This equation implies that both $(Dh_0)^2$ and $\triangle_{\mathrm{X}}h_0$ depend on x^μ only through h_0, i.e., can be regarded as functions of h_0.

Next, we analyse (A.3c) and (A.3e), which can be now written

$$R(\mathrm{X}_4)-\frac{2}{h}\triangle_{\mathrm{X}}h_0+\left(\frac{9}{8h^2}-\frac{\Phi_0^2}{2}\right)(Dh_0)^2-\frac{\partial_z^2 h_1}{h^{3/2}}+\frac{(\partial_z h_1)^2}{2h^{5/2}}=\frac{4m^2}{3}e^{2\phi}h^{1/2}, \tag{A.19a}$$
$$-\frac{\triangle_{\mathrm{X}}h_0}{2h}-\frac{\partial_z^2 h_1}{h^{3/2}}+\frac{(\partial_z h_1)^2}{h^{1/2}}\left(\frac{5}{4h^2}-\frac{\Phi_1^2}{2}\right)=\frac{m^2}{3}h^{1/2}e^{2\phi}. \tag{A.19b}$$

Here, note that the first equation and the above argument imply that $R(\mathrm{X}_4)$ can be regarded as a function of h_0. Further, by eliminating e^ϕ from these equations, we obtain

$$h^{3/2}A+h^{1/2}B+3\partial_z^2 h_1+\frac{9}{2}(\partial_z h_1)^2\left[\frac{p(h_0-h_1)+2q}{ph_1^2-2qh_1+s}\right]=0, \tag{A.20}$$

where A and B are defined by

$$A=R(\mathrm{X}_4)-\frac{9p}{8}\frac{(Dh_0)^2}{ph_0^2+2qh_0+s},\quad B=\frac{9(Dh_0)^2}{4}\frac{ph_0+q}{ph_0^2+2qh_0+s}. \tag{A.21}$$

Note that A and B can be regarded as functions of h_0 from the above arguments.

Differentiating (A.20) by y twice, we get

$$-B + h(3A + 4\partial_{h_0} B) + h^2(6\partial_{h0} A + 4\partial_{h_0}^2 B) + h^3 \partial_{h_0}^2 A = 0. \tag{A.22}$$

Since the left-hand side of this equation is a polynomial of h_1, the coefficients of powers of h should vanish separately. This requires $A = B = 0$. Hence, (A.20) is equivalent to

$$p = 0, \quad q(Dh_0)^2 = 0, \quad R(\mathrm{X}_4) = 0, \quad \partial_z^2 h_1 + \frac{3q(\partial_z h_1)^2}{s - 2qh_1} = 0. \tag{A.23}$$

This equation implies that $q = 0$ if $(Dh_0)^2 \neq 0$. On the other hand, in the case of $(Dh_0)^2 = 0$, which requires $\triangle_{\mathrm{X}} h_0 = 0$, (A.19a) reduces to

$$-h\partial_z^2 h_1 + \frac{1}{2}(\partial_z h_1)^2 = \frac{4m^2}{3} h^3 e^{2\phi}. \tag{A.24}$$

The left-hand side of this equation is linear in h_0. Hence, taking account of (A.14), we find that there exists a solution for h_1 only when $q = 0$ also in the case of $(Dh_0)^2 = 0$.

Thus, we can assume that $p = q = R(\mathrm{X}_4) = 0$. Then, (A.14) and the last equation of (A.23) reduce to

$$e^{\mp 2\phi/3} = \frac{2cs}{h}, \qquad \partial_z h_1 = k. \tag{A.25}$$

Inserting these expressions into (A.19b), we obtain

$$e^{2\phi/3} = \frac{2cs}{h}, \quad \frac{k^2}{(2cs)^3} = \frac{8m^2}{3}, \quad \triangle_{\mathrm{X}} h_0 = 0. \tag{A.26}$$

Hence, (A.16) reduces to $R_{\mu\nu}(\mathrm{X}_4) = 0$ and $D_\mu D_\nu h_0 = 0$.

To summarise, under the conditions (A.1) and (A.2), the most general solution of the field equations (4.10) is given by

$$R_{\mu\nu}(\mathrm{X}_4) = 0, \qquad h(x,z) = h_0(x) + kz, \qquad e^{2\phi} = l^2 h^{-3}, \tag{A.27}$$

where l is a constant, and h_0 and k satisfy the conditions

$$D_\mu D_\nu h_0 = 0, \qquad k^2 = \frac{8}{3} m^2 l^2. \tag{A.28}$$

Here, we can set $l = 1$ by redefining k, z and $ds^2(\mathrm{X}_4)$. Further, a solution with $(Dh_0)^2 \neq 0$ exists only when X_4 is locally flat, and in that case, h_0 can be written as a linear combination of the Minkowski coordinates [2]. This solution with Minkowskian X_4 is identical to the solution found by Chen et al. [11].

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